Mobile Geometric Graphs: Detection, Isolation and Percolation

Perla Sousi [1]

Based on joint works with

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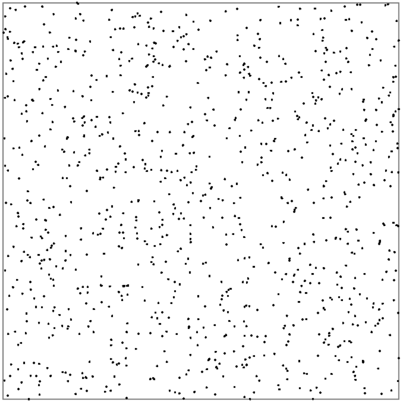
Random Geometric Graph (Boolean Model)

Nodes: Poisson point process in $\mathbb{R}^d$, intensity λ

Edges: $\exists\ (u,v) \iff d(u,v) \leq r$ (*intersecting balls of radius* $r/2$)



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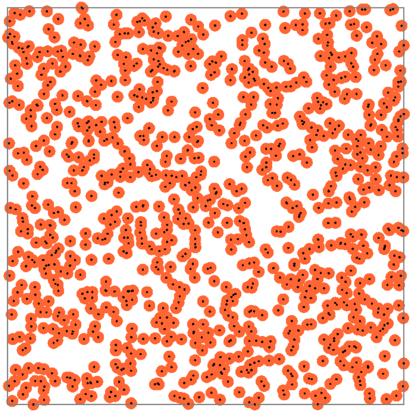

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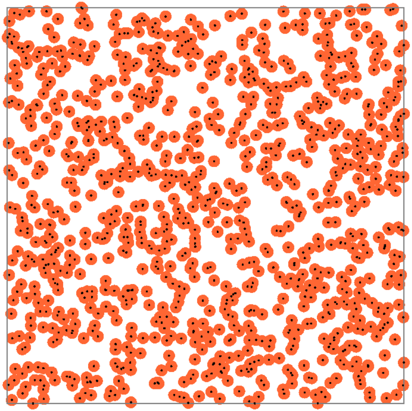

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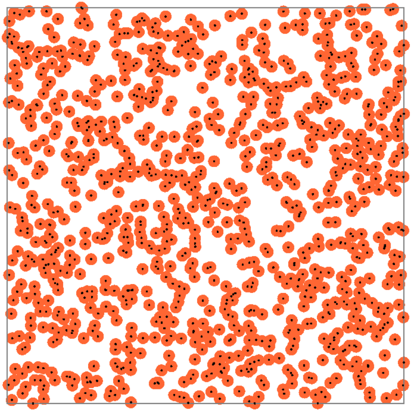

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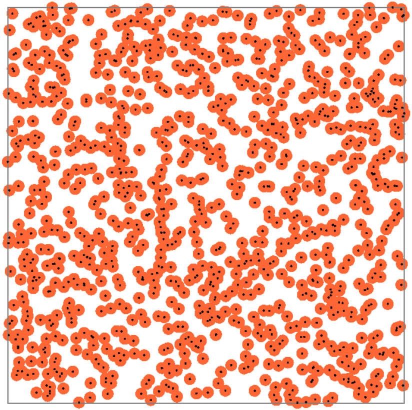

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Detection in a mobile geometric graph

Target particle initially at origin

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Want to study $\mathbb{P}(T_{\text{det}} > t)$

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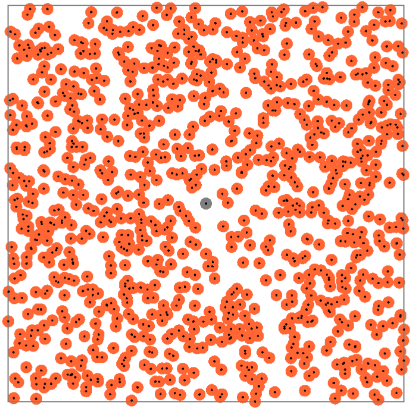

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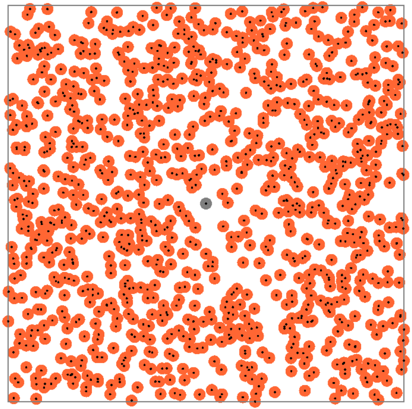

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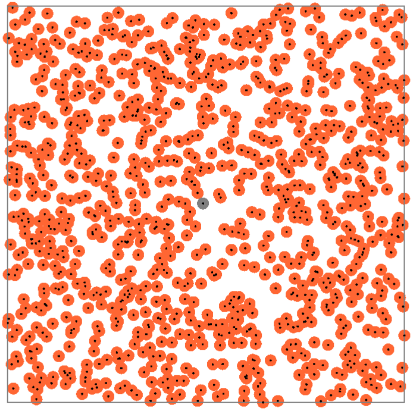

time $T_{\rm det}$

Detection of a non-mobile target

Lemma (Classical result of stochastic geometry)

Let ξ be a standard Brownian motion and $W_r(t) = \cup_{s \le t} \mathcal{B}(\xi(s), r)$, the Wiener sausage up to time t. Then

$$\mathbb{P}(T_{\mathrm{det}} > t) = \exp(-\lambda \mathbb{E}[\mathrm{vol}(W_r(t))]).$$

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In dimension 3 and above as $t \to \infty$

$$\mathbb{P}(T^f_{\mathrm{det}} > t) \leq \exp\left(-\lambda\alpha(d) c_d r^{d-2} t(1+o(1))\right).$$

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Theorem (Peres, S. (2011))

Let $(\xi(s))_s$ be a standard Brownian motion in d dimensions and f a deterministic function, $f : \mathbb{R}_+ \to \mathbb{R}^d$. Then for all $r > 0$ and all t we have that

$$\mathbb{E}[\mathrm{vol}(\cup_{s \le t} \mathcal{B}(\xi(s) + f(s), r))] \ge \mathbb{E}[\mathrm{vol}(\cup_{s \le t} \mathcal{B}(\xi(s), r))].$$

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Connection to $\mathbb{E}[\mathrm{vol}(\text{sausage around the walk})]$ and Donsker's invariance principle give the result.

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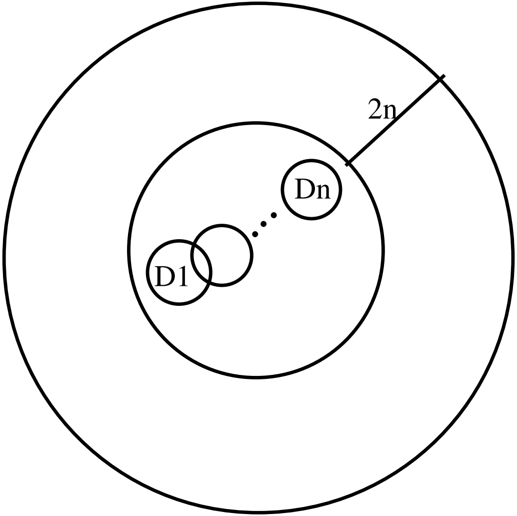
Points outside this ball cannot have reached any of the sets (D_i) by time n.



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Theorem (Special case of *Brascamp, Lieb, Luttinger (1974)*)

Let $A_1, \ldots, A_n \subset \mathbb{R}^d$ of finite volume and $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ a nonincreasing function of distance. Then

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{0 \le i \le n} \mathbb{1}(x_i \in A_i) \prod_{1 \le i \le n} \psi(x_{i-1}, x_i)\, dx_0 \ldots dx_n$$

$$\le \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{0 \le i \le n} \mathbb{1}(x_i \in A_i^*) \prod_{1 \le i \le n} \psi(x_{i-1}, x_i)\, dx_0 \ldots dx_n.$$

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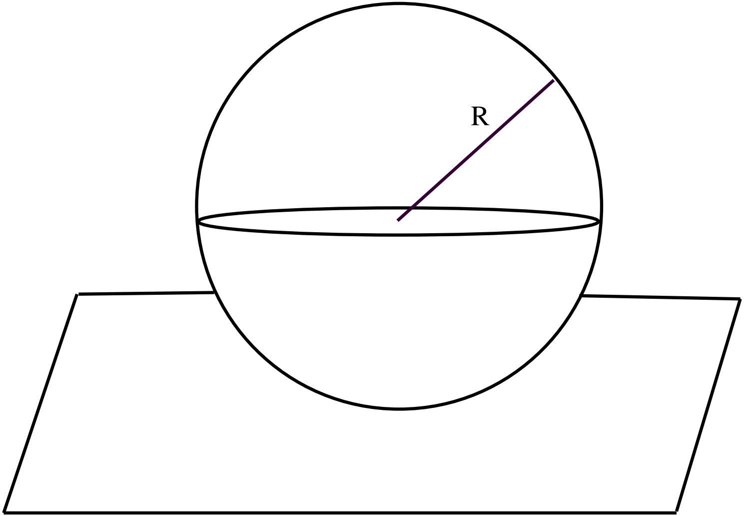

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$$\mathbb{P}(T^D_{\mathrm{det}} > n) \leq \mathbb{P}(T^C_{\mathrm{det}} > n),$$

where C_k is a cap centered at $(0, \dots, 0, -R)$ with $\mu(C_k) = \mu(D_k)$.

More general result

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Theorem (Peres, S. (2011))

Let $(\xi(s))_{s\geq 0}$ be a standard Brownian motion in $d \geq 1$ dimensions and let $(D_s)_{s\geq 0}$ be open sets in $\mathbb{R}^d$ with $\mathrm{vol}(D_s) = c$ for all s. Then for all t we have that

$$\mathbb{E}\left[\mathrm{vol}\left(\cup_{s\leq t}\left(\xi(s) + D_s\right)\right)\right] \geq \mathbb{E}\left[\mathrm{vol}\left(\cup_{s\leq t}\mathcal{B}(\xi(s), r)\right)\right],$$

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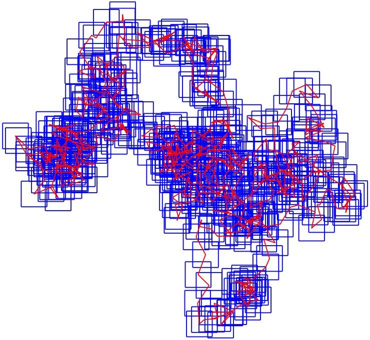
In particular this gives that the expected volume of the Wiener sausage with squares is bigger than the expected volume with balls.

Squares vs disks



Wiener sausage with squares

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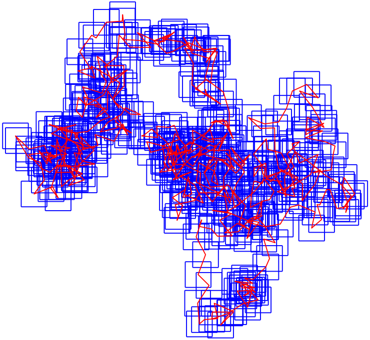

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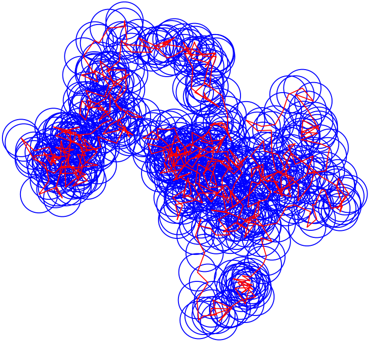

Wiener sausage with disks

A connection to capacity

Spitzer and Whitman(1964) proved that in $d \geq 3$, if $A \subset \mathbb{R}^d$ is an open set with finite volume, then

$$\frac{\mathbb{E}[\mathrm{vol}(\cup_{s \leq t}(\xi(s) + A))]}{t} \to \mathrm{Cap}(A) \text{ as } t \to \infty.$$

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Our theorem is a refinement of a classical inequality due to Pólya and Szëgo:

In $d \geq 3$ among all open sets of fixed volume, the ball has the smallest Newtonian capacity.

Isolation in a mobile geometric graph

Target particle initially at origin

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$T_{\mathrm{isol}} =$ 1st time no node is within distance r of target

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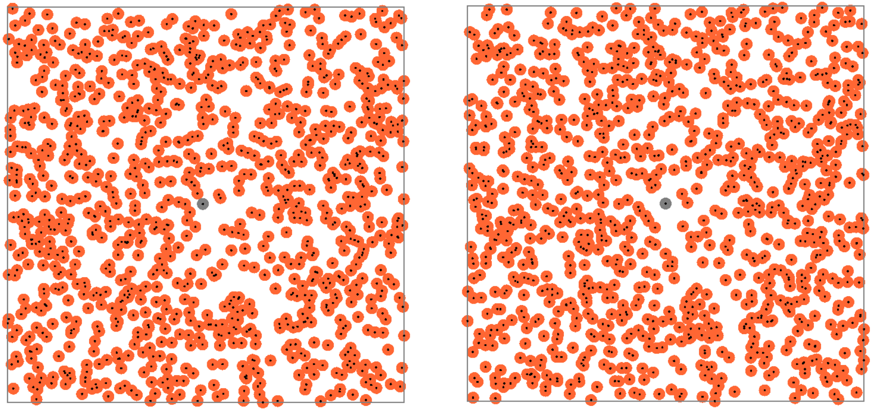

time 0 time $T_{\rm isol}$

Isolation in a mobile geometric graph

Define

$$\Psi_d(t) = \begin{cases} \sqrt{t}, & \text{for } d = 1 \\ \log t, & \text{for } d = 2 \\ 1, & \text{for } d \geq 3. \end{cases} \tag{1}$$

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Theorem (Peres, S., Stauffer (2011))

For all $d \geq 1$ as $t \to \infty$

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Lower bound matches upper bound in $d \geq 3$ and up to logarithmic factors in the exponent in $d = 2$.

Better lower bound in dimension 1

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Theorem (Peres, S., Stauffer (2011))

For $d = 1$ as $t \to \infty$

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Isolation for a mobile target

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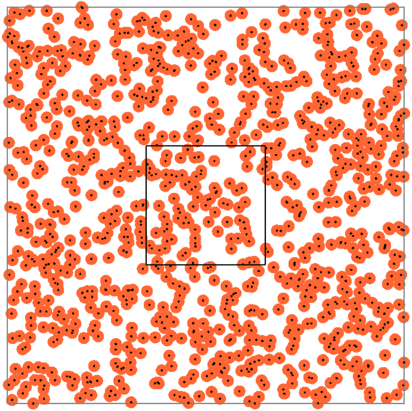
Proof uses rearrangement inequalities of Brascamp, Lieb, Luttinger ('74) and a new decoupling idea.

Coverage

Q_R = cube of side length R

$T_{\text{cov}}(Q_R)$ = 1st time all points of Q_R have been detected

Open problem proposed in Konstantopoulos'09.



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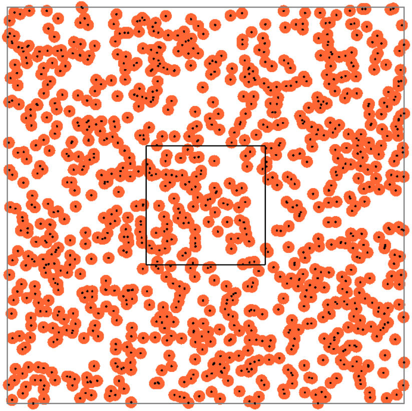

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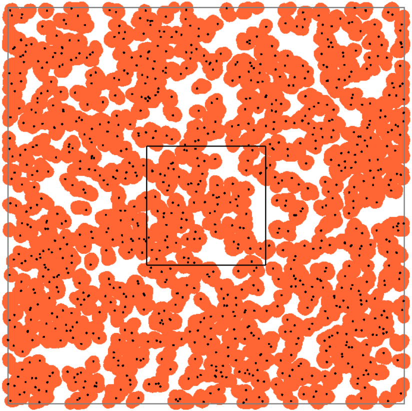

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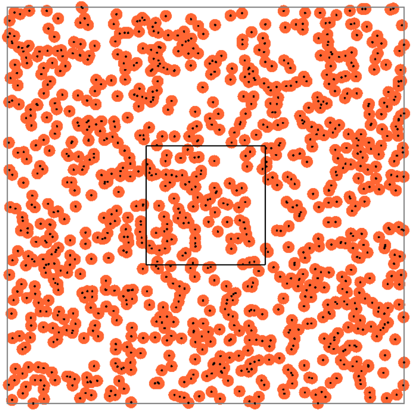

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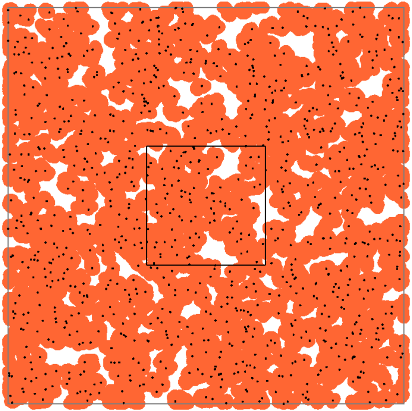

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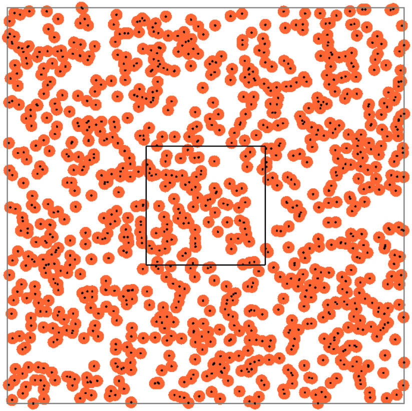

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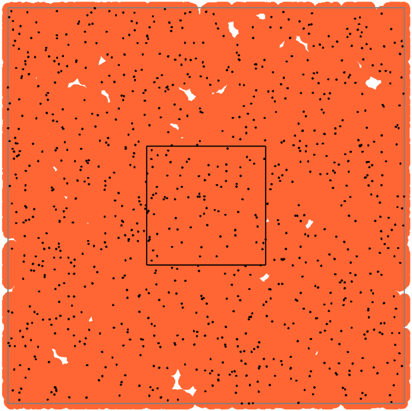

time $T_{\mathrm{cov}}(Q_R)$

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Theorem (Peres, Sinclair, S., Stauffer (2010))

As $R \to \infty$, we have that

$$\mathbb{E}\, T_{\mathrm{cov}}(Q_R) \sim \frac{2}{2\pi\lambda} \log R \log\log R \quad \textit{and} \quad \frac{T_{\mathrm{cov}}(Q_R)}{\mathbb{E}\, T_{\mathrm{cov}}(Q_R)} \to 1 \textit{ in probability}$$

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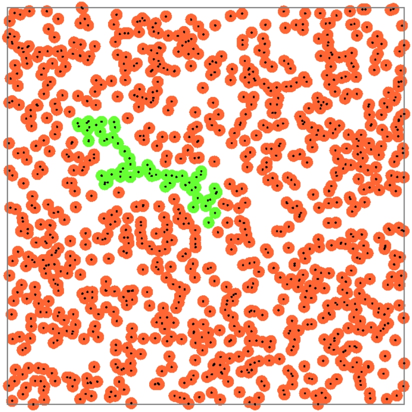
Theorem ((General result), Peres, Sinclair, S., Stauffer (2010))

For a set A and $R > 0$, let $RA = \{Ra : a \in A\}$. If A has Minkowski dimension α, then as $R \to \infty$

$$\mathbb{E}\, T_{\mathrm{cov}}(RA) \sim \frac{\alpha}{2\pi\lambda} \log R \log\log R \quad \text{and} \quad \frac{T_{\mathrm{cov}}(RA)}{\mathbb{E}\, T_{\mathrm{cov}}(RA)} \to 1 \text{ in probability}$$

Percolation on Mobile Geometric Graph

$\exists \lambda_c$ s.t. $\lambda > \lambda_c \Rightarrow$ *a.s.* $\exists$ infinite component at fixed time *s*



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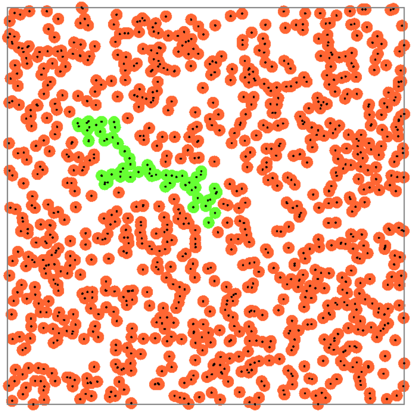

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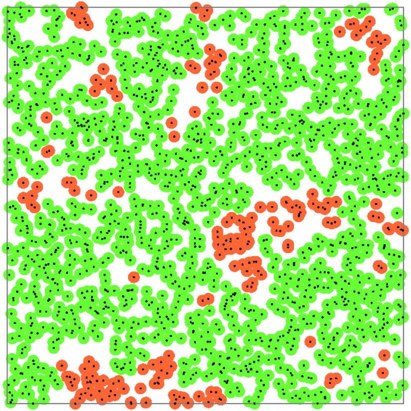

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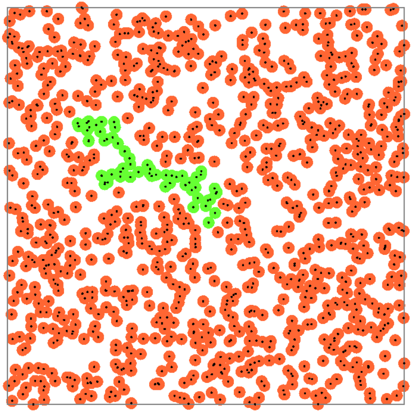
Percolation on Mobile Geometric Graph

$\lambda > \lambda_c \Rightarrow$ *a.s.* $\exists$ infinite component for every s (van den Berg, Meester, White'97)



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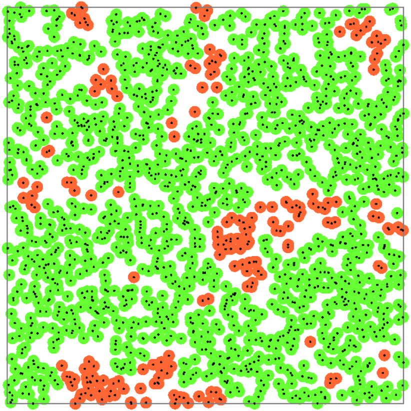

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Percolation

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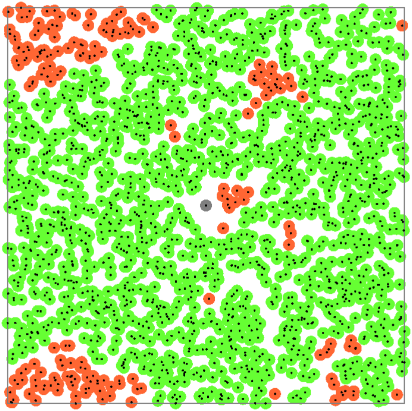
We assume $\lambda > \lambda_c$

T_{perc} = 1st time target belongs to infinite component

Want to study $\mathbb{P}(T_{\text{perc}} > t)$



time 0

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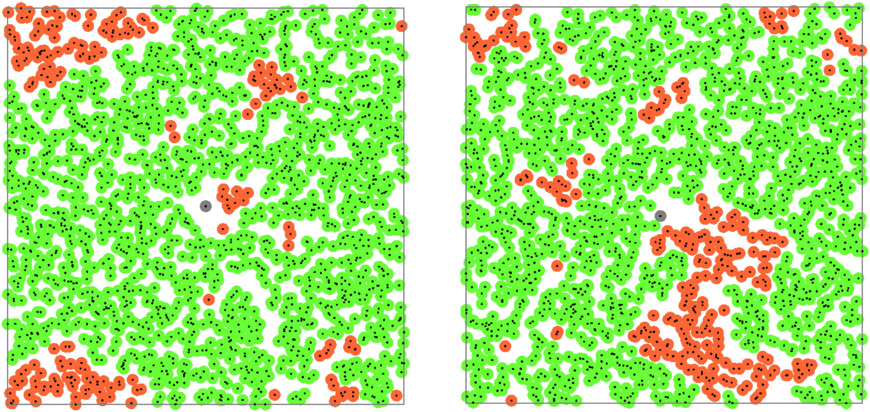

time 0 | time T_{perc}

Percolation

Lower bound in discrete time via FKG (extends to continuous time):

$\mathbb{P}(\text{target} \notin \text{infinite component at time } s) = \mathbb{P}(T_{\mathrm{perc}} > 0) = e^{-c}$

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Lower bound via detection:

$\mathbb{P}(T_{\text{perc}} > t) \geq \mathbb{P}(T_{\text{det}} > t) \geq e^{-c'' t / \log t}$

Percolation

Upper bound:

$\mathbb{P}(T_{\rm perc} > t) \leq \exp\left(-c\sqrt{t}\right)$ (Sinclair, Stauffer 2010)

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Before starting the proof...

Observe graphs in discrete time steps only $i = 0, 1, 2, \ldots$

$H_i = \{\text{target} \notin \text{infinite component at step } i\}$

$\mathbb{P}(T_{\mathrm{perc}} > t) \le \mathbb{P}(\cap_{i=0}^{t} H_i)$

- Y. Peres, A. Sinclair, P. Sousi and A. Stauffer
  Mobile geometric graphs: detection, coverage and percolation.
  to appear in *PTRF*.
- Y. Peres and P. Sousi.
  An isoperimetric nequality for the Wiener sausage.
  to appear in *GAFA*.
- Y. Peres, P. Sousi and A. Stauffer.
  The Isolation Time of Poisson Brownian motions.
  Arxiv e-prints.