Reflecting Brownian motion in two dimensions: Exact asymptotics for the stationary distribution

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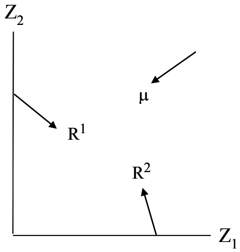
Outline

- Background
- Geometric objects
- Theorems
- Proof techniques

Part I: Background

Semimartingale reflecting Brownian motion (SRBM)

$$Z(t) = X(t) + RY(t) \quad \text{for all } t \geq 0,$$
X is a (μ, Σ) Brownian motion,
$Z(t) \in \mathbb{R}^n_+$ for all $t \geq 0$,
$Y(\cdot)$ is continuous and nondecreasing with $Y(0) = 0$,
$Y_i(\cdot)$ only increases when $Z_i(\cdot) = 0, \quad i = 1, \ldots, n.$



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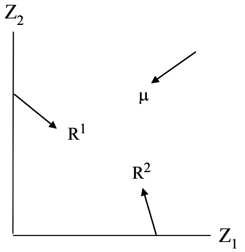
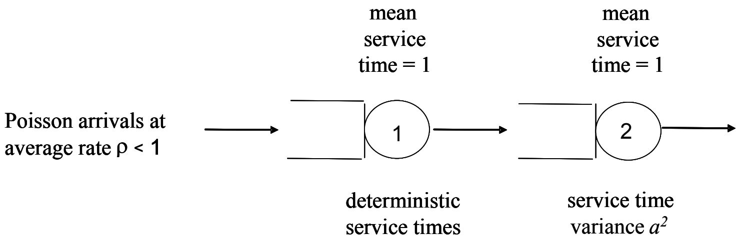


- We focus on $n = 2$.
- R is assumed to be completely-$\mathcal{S}$

$$R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}.$$

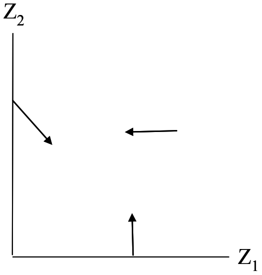
- Z exists and is unique in distribution; (Taylor-Williams 92)

A queueing network



Reiman (84), Harrison-Reiman (81)

Its Approximating SRBM



$$Z_1(t) = X_1(t) + Y_1(t),$$
$$Z_2(t) = X_2(t) - Y_1(t) + Y_2(t)$$
drift of X is $\mu = (\rho - 1, 0)$,

covariance of X is $\Sigma = \begin{pmatrix} \rho & 0 \\ 0 & a^2 \end{pmatrix}$

- R. J. Williams (95), Semimartingale reflecting Brownian motions in the orthant, in Stochastic Networks, eds. F. P. Kelly and R. J. Williams, the IMA Volumes in Mathematics and its Applications, Vol. 71 (Springer, New York)
- R. J. Williams (96), On the approximation of queueing networks in heavy traffic, in Stochastic Networks: Theory and Applications, eds. F. P. Kelly, S. Zachary and I. Ziedens, Royal Statistical Society (Oxford Univ. Press, Oxford)

Stationary distribution of an SRBM

Proposition (Hobson-Rogers 94, Harrison-Hasenbein 09)

Assume that

$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \tag{1}$$

$$r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad \text{and} \quad r_{11}\mu_2 - r_{21}\mu_1 < 0. \tag{2}$$

Then SRBM Z has a unique stationary distribution π.

- Condition (1): R is a $\mathcal{P}$-matrix.
- Condition (2): $R^{-1}\mu < 0$.

Tail asymptotics of the stationary distribution

- Let $Z(\infty) = (Z_1(\infty), Z_2(\infty))$ be the random vector that has the stationary distribution π.
- For any $c \in \mathbb{R}_+^2$, set

$$\langle c, Z(\infty) \rangle = c_1 Z_1(\infty) + c_2 Z_2(\infty).$$

- We are interested in finding a function $f_c(x)$ that satisfies

$$\lim_{x \to \infty} \frac{\mathbb{P}\{\langle c, Z(\infty) \rangle > x\}}{f_c(x)} = b$$

  for some constant $b > 0$.
- The function $bf_c(x)$ is said to be the exact asymptotic of $\mathbb{P}\{\langle c, Z(\infty) \rangle > x\}$.

An overview

Main Result

- *For an SRBM satisfying (1) and (2), the exact asymptotic is given by*

$$f_c(x) = x^{\kappa_c} e^{-\alpha_c x}.$$

- *The decay rate α_c and the constant κ_c can be computed explicitly from the primitive date (μ, Σ, R).*
- *The constant κ_c must take one of the values $-3/2$, $-1/2$, 0 or 1.*

Let $p_c(x)$ be the density of $\langle c, Z(\infty) \rangle$. In most cases, we have

$$\lim_{x \to \infty} \frac{p_c(x)}{f_c(x)} = b \text{ for some } b > 0.$$

A diversion: Laplace transform

- Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and integrable function. Define

$$\tilde{f}(z) = \int_0^\infty e^{zx} f(x)dx, \quad \Re z < \alpha_f,$$

where $\alpha_f = \sup\{\theta \geq 0 : \tilde{f}(\theta) < \infty\}$. $\tilde{f}$ is analytic in $\Re z < \alpha_f$.

- When f is a Gamma(k, α) density, namely,

$$f(x) = \frac{c}{\Gamma(k)} x^{k-1} e^{-\alpha x},$$

then $\alpha_f = \alpha$ and

$$\tilde{f}(z) = \frac{c}{(\alpha - z)^k} \equiv g(z) \quad \text{for } \Re z < \alpha.$$

- Complex function g is an "analytic" extension of $\tilde{f}$.

A diversion: complex inversion

Proposition (When k is a positive integer)

Suppose that the analytic extension g of $\tilde{f}$ satisfies that

$$g(z) - \frac{c_0}{(\alpha_0 - z)^k}$$

is analytic for $\Re z < \alpha_1$ for some $\alpha_1 > \alpha_0$. Then, under some mild conditions on g,

$$f(x) \sim \frac{c_0}{\Gamma(k)} x^{k-1} e^{-\alpha_0 x} \quad \text{as } x \to \infty. \tag{3}$$

When α_0 is the kth order pole of $g(z)$, then (3) holds.

Convergence domain

- Define

$$\varphi(\theta_1, \theta_2) = \mathbb{E}\Big[e^{\langle \theta, Z(\infty) \rangle}\Big].$$

Then the transform of $\langle c, Z(\infty) \rangle$ has the following expression

$$\psi_c(z) = \mathbb{E}\big(e^{z\langle c, Z(\infty)\rangle}\big) = \varphi(zc).$$

- Define the convergence domain of φ

$$\mathcal{D} = \text{ interior of } \{\theta \in \mathbb{R}^2 : \varphi(\theta) < \infty\}.$$

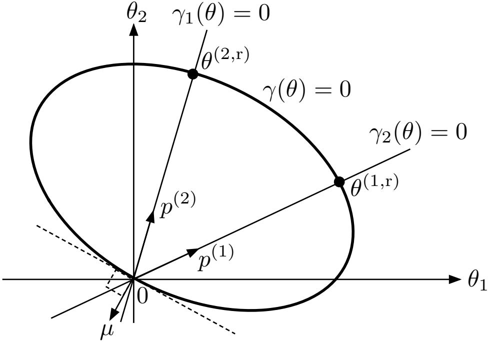
Some literature

- Fayolle and Iasnogorodski (1979),
- Cohen and Boxma (1983), Cohen (1992), Fayolle, Iasnogorodski and Malyshev (99): Riemann, Riemann-Hilbert, Wiener-Hopf problem; "exact" analysis and two-dimensional.
- Lieshout and Mandjes (07, 08), Miyazawa and Rolski (09),
- Miyazawa and Kobayashi (2011)

Part II: Geometric objects

Geometric objects: an ellipse and two lines

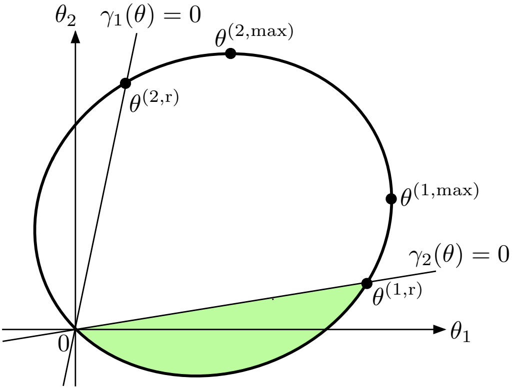
$$\gamma(\theta) = -\langle \theta, \mu \rangle - \frac{1}{2}\langle \theta, \Sigma\theta \rangle,$$
$$\gamma_1(\theta) = r_{11}\theta_1 + r_{21}\theta_2 = \langle R^1, \theta \rangle,$$
$$\gamma_2(\theta) = r_{12}\theta_1 + r_{22}\theta_2 = \langle R^2, \theta \rangle,$$



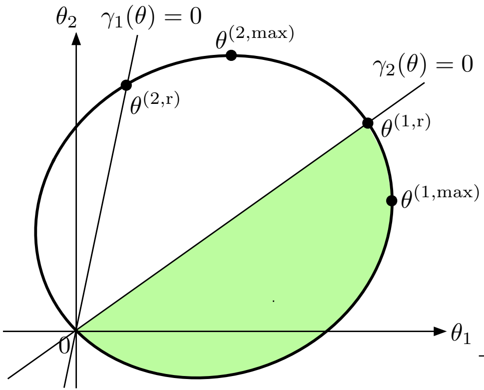
Stability condition (2): $\langle \mu, p^{(1)} \rangle < 0, \quad \langle \mu, p^{(2)} \rangle < 0.$ (4)

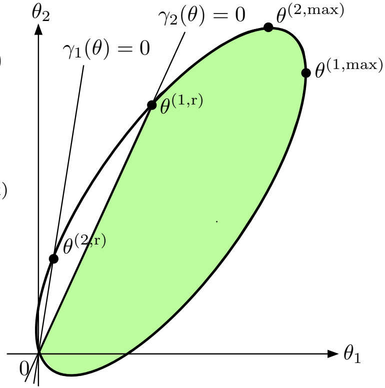
Domain Γ_1 is shaded; $\theta^{(1,\max)} \notin \partial\Gamma_1$; $\theta^{(1,\Gamma)} = \theta^{(1,r)}$

$$\Gamma = \{\theta \in \mathbb{R}^2 : \gamma(\theta) > 0\}, \qquad \Gamma_1 = \{\theta \in \Gamma : \gamma_2(\theta) < 0\}.$$



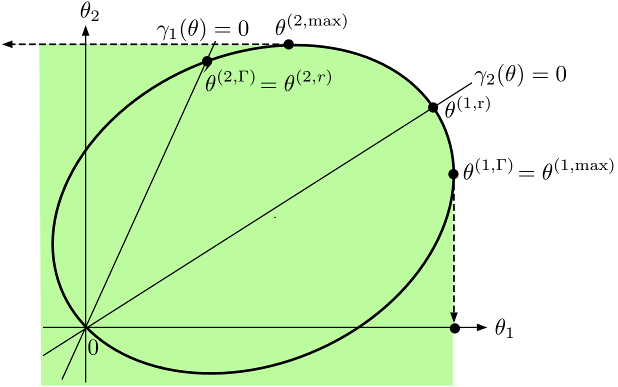
Domain Γ_1; $\theta^{(1,\max)} \in \partial\Gamma_1$; $\theta^{(1,\Gamma)} = \theta^{(1,\max)}$



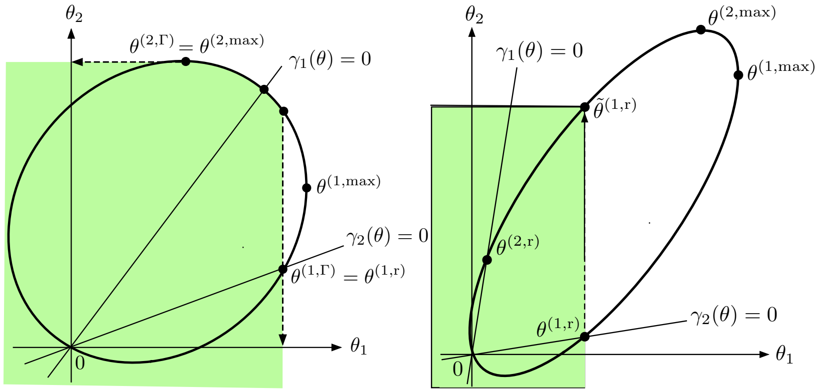


Part III: Theorems

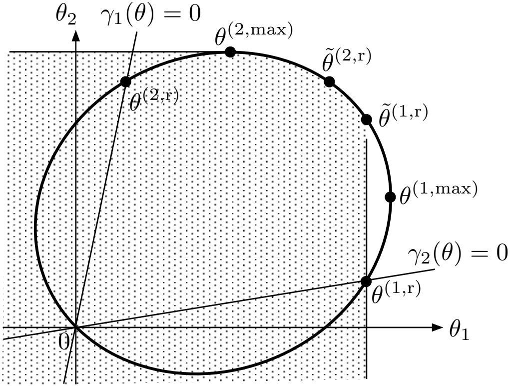
$$\mathcal{D}^{(1)} = \left\{\theta \in \mathbb{R}^2;\ \text{there exists a } \theta' \in \Gamma \text{ such that } \theta < \theta' \text{ and } \theta'_1 < \theta_1^{(1,\Gamma)}\right\}.$$

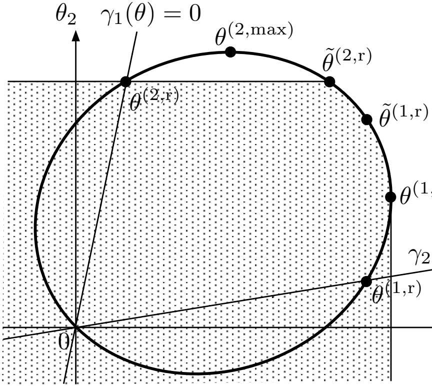


Domain $\mathcal{D}_1$; $\theta^{(1,\max)} \notin \partial\Gamma_1$



Domains $\mathcal{D}_1$ and $\mathcal{D}_2$

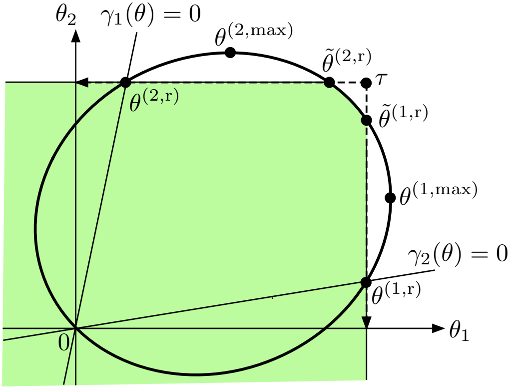




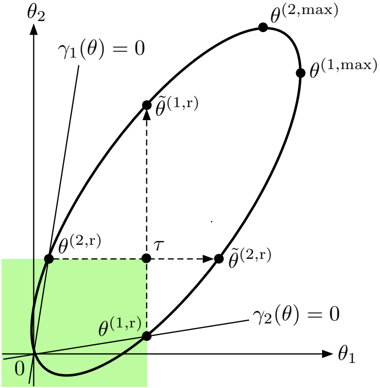
Theorem 1 when $\tau \notin \partial\Gamma$ (Category I); τ is outside of $\bar{\Gamma}$

Theorem

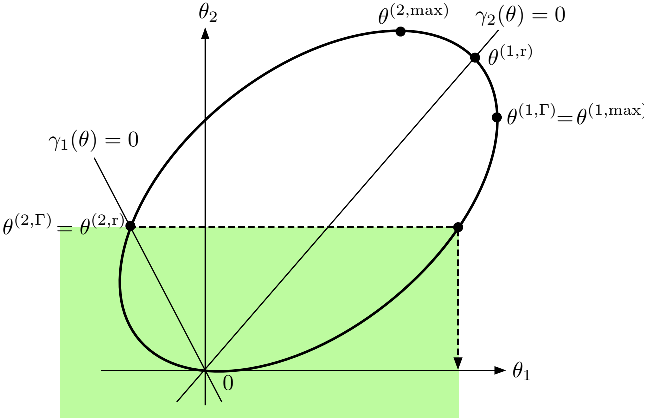
$$\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$$



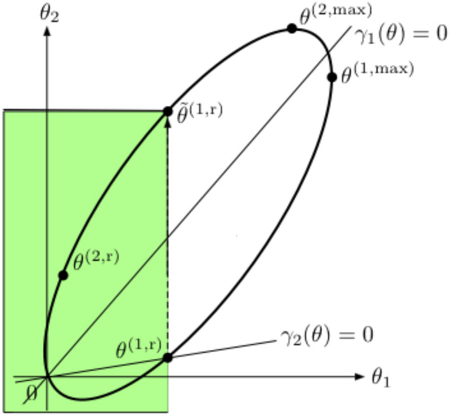
Theorem 1 when $\tau \notin \partial\Gamma$ (Category I); τ is inside of Γ



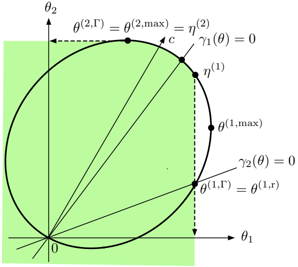
Theorem 1 when $\tau \in \partial\Gamma$; $\theta^{(2,\Gamma)} \le \theta^{(1,\Gamma)}$ (Category II)



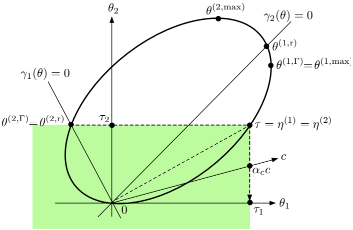
Theorem 1 when $\tau \in \partial\Gamma$; $\theta^{(1,\Gamma)} \leq \theta^{(2,\Gamma)}$ (Category III)

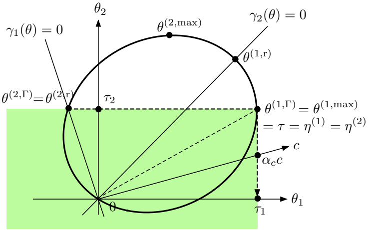


Theorem 2 (illustrated)



$$f_c(x) = \begin{cases} e^{-\alpha_c x} & \text{if } \beta \in [0, \beta_1), \\ x e^{-\alpha_c x} & \text{if } \beta = \beta_1, \\ e^{-\alpha_c x} & \text{if } \beta \in (\beta_1, \beta_2), \\ x^{-1/2} e^{-\alpha_c x} & \text{if } \beta = \beta_2 \\ x^{-3/2} e^{-\alpha_c x} & \text{if } \beta \in (\beta_2, \pi/2] \end{cases}, \quad \alpha_c = \begin{cases} \tau_1/c_1 & \text{if } \beta < \beta_1 \\ z_c & \text{if } \beta \in [\beta_1, \beta_2] \\ & \text{where } z_c c \in \partial\Gamma \\ \tau_2/c_2 & \text{if } \beta > \beta_2 \end{cases}$$





When $\beta \in [0, \beta_1)$

$$f_c(x) = \begin{cases} xe^{-\alpha_c x} & \text{if } \tau = \theta^{(1,r)} \neq \theta^{(1,\max)} \\ x^{-1/2}e^{-\alpha_c x} & \text{if } \tau = \theta^{(1,\max} \neq \theta^{(1,r)}, \\ e^{-\alpha_c x} & \text{otherwise,} \end{cases}$$

When $\beta \geq \beta_1$,

$$f_c(x) = \begin{cases} xe^{-\alpha_c x} & \text{if } \beta = \beta_1, \\ e^{-\alpha_c x} & \text{if } \beta > \beta_1. \end{cases}$$

Part 4: Proof techniques

Basic adjoint relationship

- For each $f \in C_b^2(\mathbb{R}_+^2)$, let

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{2} \Sigma_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{2} \mu_i \frac{\partial f}{\partial x_i}(x),$$

$$D_i f(x) = \langle R^i, \nabla f \rangle.$$

- The following holds for each $f \in C_b^2(\mathbb{R}_+^2)$:

$$\int_{\mathbb{R}_+^2} Lf(x)\pi(dx) + \sum_{i=1}^{2} \int_{\mathbb{R}_+^2} D_i f(x)\nu_i(dx) = 0, \tag{5}$$

where $\nu_i(A) = \mathbb{E}_\pi\left[\int_0^1 1_{\{Z(u)\in A\}}\, dY_i(u)\right]$ defines the ith boundary measure.

- Fix $\theta = \mathbb{R}^2$. Ignoring the $C_b^2(\mathbb{R}_+^2)$ assumption, applying BAR (5) to function

$$f(x_1, x_2) = e^{\theta_1 x_1 + \theta_2 x_2} = e^{\langle \theta, x \rangle}. \tag{6}$$

A key relationship

- Let φ_1 and φ_2 be boundary Laplace transforms:

$$\varphi_1(\theta_2) = \mathbb{E}_\pi \left[\int_0^1 e^{\theta_2 Z_2(u)}\, dY_1(u) \right], \varphi_2(\theta_1) = \mathbb{E}_\pi \left[\int_0^1 e^{\theta_1 Z_1(u)}\, dY_2(u) \right]$$

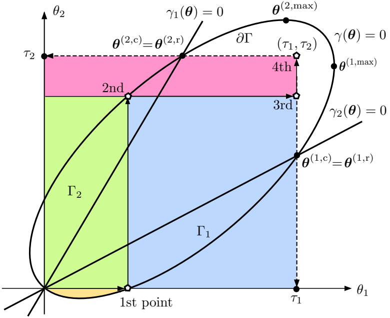
- Recall $\varphi(\theta_1, \theta_2) = \mathbb{E}_\pi \left[e^{\theta_1 Z_1(\infty) + \theta_2 Z_2(\infty)} \right]$.
- If $e^{\theta_1 x_1 + \theta_2 x_2} \in C_b^2(\mathbb{R}_+^2)$, one has the key relationship

$$\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1). \tag{7}$$

- Can the $C_b^2(\mathbb{R}_+^2)$ assumption be ignored?

Key steps in proving Theorem 1

- $\varphi(\theta) < \infty$ implies $\varphi_1(\theta_2) < \infty$ and $\varphi_2(\theta_1) < \infty$.
- $\theta \in \Gamma_1$ and $\varphi_1(\theta_2) < \infty$ imply $\varphi(\theta) < \infty$.
- $\theta \in \Gamma$, $\varphi_1(\theta_2) < \infty$ and $\varphi_2(\theta_1) < \infty$ imply $\varphi(\theta) < \infty$.



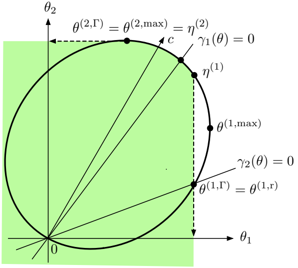
An illustration for proving Theorems 2: $\beta \in (\beta_1, \beta_2)$

- Key relationship (7) gives

$$\gamma(zc)\psi_c(z) = \gamma_1(zc)\varphi_1(c_2 z) + \gamma_2(zc)\varphi_2(c_1 z) \quad \text{for } \Re z < \alpha_c.$$

- Letting $\gamma(zc) = z\zeta_c(z)$, we have

$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$



- $\varphi_2(z)$ is analytic on $\Re z < \tau_1$.
- $\varphi_1(z)$ is analytic on $\Re z < \tau_2$.
- $\alpha_c c \in \partial\Gamma$.
- $\alpha_c < \min(\tau_1/c_1, \tau_2/c_2)$.
- $\psi_c(z)$ has a simple pole at $z = \alpha_c$.
- $f_c(x) = e^{-\alpha_c x}$.

An illustration for proving Theorems 2: $\beta < \beta_1$ and $\alpha_c c \notin \partial\Gamma$

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- Letting $\gamma(zc) = z\zeta_c(z)$, we have

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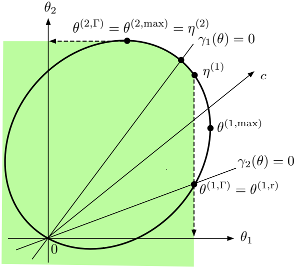
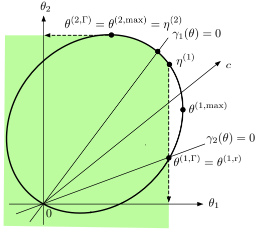


- $\varphi_2(z)$ has a simple pole at $\Re z < \tau_1$.
- $\varphi_1(z)$ is analytic on $\Re z < \tau_2$.
- $\psi_c(z)$ has a simple pole at $z = \alpha_c$.
- $f_c(x) = e^{-\alpha_c x}$.



- $(z, \underline{f}_2(z)) \in \partial\Gamma$.
- $\gamma(z, \underline{f}_2(z)) = 0$.

$$\varphi_2(z) = -\frac{\gamma_1(z, \underline{f}_2(z))\varphi_1(\underline{f}_2(z))}{\gamma_2(z, \underline{f}_2(z))}.$$

$$\underline{f}_2(\theta_1) = -\frac{\mu_2 + \Sigma_{12} z}{\Sigma_{22}} - \frac{\sqrt{\det(\Sigma)}}{\Sigma_{22}}\sqrt{\left(\theta_1 - \theta_1^{(1,\min)}\right)\left(\theta_1^{(1,\max)} - \theta_1\right)}.$$

$$\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) = \gamma_1(\theta_1, \theta_2)\varphi_1(\theta_2) + \gamma_2(\theta_1, \theta_2)\varphi_2(\theta_1).$$

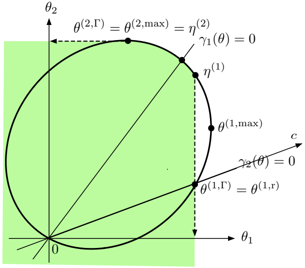
An illustration for proving Theorems 2: $\beta < \beta_1$ and $\alpha_c c \in \partial\Gamma$

- Key relationship (7) gives

$$\gamma(zc)\psi_c(z) = \gamma_1(zc)\varphi_1(c_2 z) + \gamma_2(zc)\varphi_2(c_1 z) \quad \text{for } \Re z < \alpha_c.$$

- In this case, $\gamma_2(c) = 0$. Thus, we have

$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$



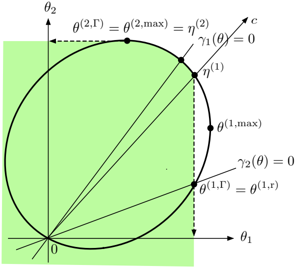
- $\varphi_1(z)$ is analytic on $\Re z < \tau_2$.
- $\psi_c(z)$ has a simple pole at $z = \alpha_c$.
- $f_c(x) = e^{-\alpha_c x}$.

- Key relationship (7) gives

$$\gamma(zc)\psi_c(z) = \gamma_1(zc)\varphi_1(c_2 z) + \gamma_2(zc)\varphi_2(c_1 z) \quad \text{for } \Re z < \alpha_c.$$

- Letting $\gamma(zc) = z\zeta_c(z)$, we have

$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$



- $\varphi_2(z)$ has a simple pole at $\Re z < \tau_1$.
- $\varphi_1(z)$ is analytic on $\Re z < \tau_2$.
- $\alpha_c c \in \partial\Gamma$.
- $\psi_c(z)$ has a double pole at $z = \alpha_c$.
- $f_c(x) = xe^{-\alpha_c x}$.

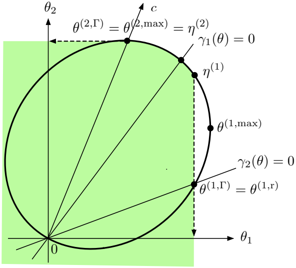
An illustration for proving Theorems 2: $\beta = \beta_2$

- Key relationship (7) gives

$$\gamma(zc)\psi_c(z) = \gamma_1(zc)\varphi_1(c_2 z) + \gamma_2(zc)\varphi_2(c_1 z) \quad \text{for } \Re z < \alpha_c.$$

- Letting $\gamma(zc) = z\zeta_c(z)$, we have

$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$



- $\varphi_2(z)$ is analytic on $\Re z < \tau_1$.
- $\varphi_1(z)$ has "-1/2 pole" at $z = \theta^{(2,\max)}$.
- α_c is a zero of $\zeta_c(z) = 0$.
- $\psi_c(z)$ has a half pole at $z = \alpha_c$.
- $f_c(x) = x^{-1/2} e^{-\alpha_c x}$.

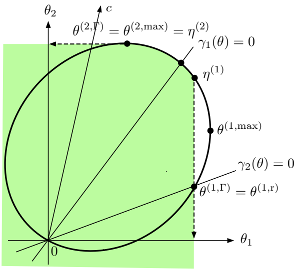
An illustration for proving Theorems 2: $\beta > \beta_2$

- Key relationship (7) gives

$$\gamma(zc)\psi_c(z) = \gamma_1(zc)\varphi_1(c_2 z) + \gamma_2(zc)\varphi_2(c_1 z) \quad \text{for } \Re z < \alpha_c.$$

- Letting $\gamma(zc) = z\zeta_c(z)$, we have

$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$



- $\varphi_2(z)$ is analytic on $\Re z < \tau_1$.
- $\varphi_1(z)$ has "$-1/2$ pole" at $z = \theta^{(2,\max)}$.
- α_c is not a zero of $\zeta_c(z) = 0$.
- $\psi_c(z)$ has a $-1/2$ pole at $z = \alpha_c$.
- $f_c(x) = x^{-3/2}e^{-\alpha_c x}$.

Singularity properties of $\varphi_2(z)$ at $z = \tau_1$

- In Category I when $\tau_1 < \theta_1^{(1,max)}$, simple pole
- In Category I when $\tau_1 = \theta_1^{(1,max)}$
  - When $\theta^{(1,max)} \neq \theta^{(1,r)}$, 1/2 "analytic" or −1/2 "pole"
  - When $\theta^{(1,max)} = \theta^{(1,r)}$, 1/2 "pole"
- In Category II when $\tau_1 < \theta_1^{(1,max)}$,
  - When $\theta^{(1,max)} \neq \theta^{(1,r)}$, simple pole
  - When $\theta^{(1,max)} = \theta^{(1,r)}$, double pole
- In Category II when $\tau_1 = \theta_1^{(1,max)}$
  - When $\theta^{(1,max)} \neq \theta^{(1,r)}$, 1/2"pole"
  - When $\theta^{(1,max)} = \theta^{(1,r)}$, 1 "pole"

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